Fioravanti: Automorphisms of Special Groups

Motivation / survey

→ given a family $\mathcal{F}$ of f.g. groups, when is $Out(G)$ "structured" for all $G \in \mathcal{F}$, and when can it be "wild"?

→ fundamental examples:

$$Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$
$$Out(\pi_1 S_g) = MCG^{\pm}(S_g)$$
$$Out(F_n).$$

Good properties for $G = F_n, \pi_1 S_g, \mathbb{Z}^n$	other good groups	failures
$Out(G)$ is fg (even VF)	• hyperbolic gps (Rips-Sela) or toral rel hyp. (GL). • polycyclic gps (Baues-Grunewald) • $\pi_1(M^3)$ (closed) • RAAGs (Lawrence-Servatius)	• Baumslag-Solitar gps. (CCL) • fp metabelian (me!)
$Out(G)$ is virtually generated by Dehn twists (in the gog sense)	• hyp, $\pi_1(M^3)$, RAAGs	• nilpotent gps • Thompson's V (Q: is $Out(V)$ fg?) • (non-compact) special gps (= arbitrary subgp of RAAG) (IMM '23) • CAT(0) (Martelli '25)
$\forall \phi \in Out(G)$, $\exists$ only fin. many growth rates + they are $n^d \lambda^n$, λ alg. integer.	• (toral rel.) hyp (CHHL) Q: $BS(n,m)$?	• $F_2 *_{\langle a \rangle \cong \mathbb{Z} = \langle b \rangle} BS(1,2)$ $a \to a$, $b \to ba$ on F_2, amalgamated to an exp. distorted has autos growing $\sim \log n$ • (Coulon '22) $\exists$ fg. small cancellation gps with autos of intermediate growth.

General goal: transfer properties from negative curvature to non-positive curvature.

def A group G is special if

- $G \cong \pi_1(C)$, C compact special ($\Rightarrow$ npc) cube complex (Haglund-Wise '08)
- equivalently: G can be embedded into a RAAG as a convex-cocompact subgp

(ie: G stabilises a convex subcx Y of the univ. cover X_Γ of the Salvetti complex of A_Γ, and $G \curvearrowright Y$ is cocompact)

Thm 1 : G special $\Rightarrow$ $Out(G)$ is f.g.

(also true for relative automorphism groups $Out(G, \mathcal{H}^t)$, where $\mathcal{H}$ is a finite set of f.g. subgps)

Questions : (1) is $Out(G)$ fp? Type VF?

(2) G virtually special?

(3) CAT(0) gps?

cocompactly cubulated? [Rips]

non-compact special?

Thm 2 : G special $\Rightarrow \exists\, G_0 \trianglelefteq G$ finite index st $Out(G_0)$ is virtually generated by Dehn twists.

- $\forall\, H$ hyp. 3-fld gp, $\exists$ finite index subgp $G < \mathbb{Z}^2 \times H$ st $Out(G)$ is not virtually generated by Dehn twists.

Thm 3 G virtually special $\Rightarrow$ $Out(G)$:

- is residually finite (Antolín - Minasyan - Sisto
- has finite vcd
- satisfies the Tits alternative.

Thm 4 G special, $\phi \in Out(G)$

(a) The stretch factor $\lambda_\phi := \max_{g \in G} \lim \|\phi^n(g)\|^{1/n} \geq 1$

(b) $\lambda_\phi = 1 \Rightarrow \phi$ grows at most polynomially

(c) if ϕ is coarse-median preserving, then ϕ has only finitely many growth rates $\sim n^p \lambda^n$, $\lambda > 1$, and all the others are at most polynomial.

Question: $\phi \in Out(A_\Gamma)$, untwisted ($\Leftrightarrow$ cmp). Are at-most polynomial growth rates exactly polynomial.

Automorphisms of special groups - 2

Thm G special $\Rightarrow$ $\mathrm{Out}(G)$ is f.g.

general strategy: $\phi_n \in \mathrm{Out}(G)$ infinite sequence $\rightsquigarrow$ $G \curvearrowright X$ "degeneration" $\rightsquigarrow$ $G \curvearrowright T$ simplicial tree with Dehn twists st $\phi_n \tau_n$ is "simpler" than ϕ_n.

↑ "Bestvina - Paulin"

↑ "Rips machine"

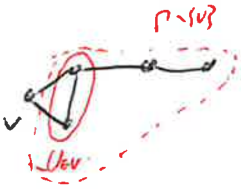
Setup

- G special group
- $i : G \hookrightarrow A_\Gamma$ convex-cocompact embedding into a RAAG. (fixed for the whole time, identifying $G \cong i(G)$)
- X_Γ universal cover of the Salvetti complex of A_Γ.
- $v \in \Gamma \rightsquigarrow A_\Gamma = A_{\Gamma \setminus \{v\}} *_{A_{\mathrm{lk}(v)}}$ HNN splitting

  $A_\Gamma \curvearrowright T^v$ Bass-Serre tree of this splitting

(equivalently, tree dual to family of hyperplanes labelled by $v \in \Gamma$.)

eg. $\Gamma =$



Rem $v \in \Gamma \rightsquigarrow \pi^v : X_\Gamma \to T^v$ is a natural A_Γ-equivariant, 1-Lipschitz, median-preserving collapse map.

$\rightsquigarrow \prod_{v\in\Gamma} \pi^v : X_\Gamma \hookrightarrow \prod_{v\in\Gamma} T^v$ equivariant, ℓ^1-isometric embedding

(nb: these trees are not at all canonical; they are determined by the choice of embedding at the start)

Important subgps:

$H \leq G$ is

- convex-cocompact if H acts cocompactly on a convex subcomplex of X_Γ.
- a centraliser in G if $H = Z_G(Z_G(H))$ (nb: $H \leq Z_G Z_G(H)$ always)

  equivalently: $\exists\, \Omega \subseteq G$ finite subset st $H = Z_G(\Omega)$.

Bestvina-Paulin construction

start with :

- $\varphi_n \in Aut(G)$ sequence with pairwise distinct projections to $Out(G)$
- $S \subseteq G$ finite generating set
- $\omega \subseteq 2^{\mathbb{N}}$ non-principal ultrafilter

$\rightsquigarrow \lambda_n := \min_{x \in X_n} \max_{s \in S} d(x, \varphi_n(s)x)$

$\rightsquigarrow$ $p_n \in X_n$ points realising this minimum

Exercise : $\lambda_n \xrightarrow[n \to +\infty]{} +\infty$

Def the degeneration given by (φ_n) (and by S, p_n, ω, i) is the ultralimit

$$G \curvearrowright X_\omega := \lim_\omega (G \underset{\substack{\uparrow \\ g \cdot x = \varphi_n(g)x}}{\curvearrowright} \tfrac{1}{\lambda_n} X_n , p_n)$$

Remarks

1) as a metric space, X_ω is an asymptotic cone of X_n, hence a finite dimensional median space.

2) $G \curvearrowright X_\omega$ has no global fixed points.

- S (easy; take word lengths of one in the other); p_n (hard) do not change $G \curvearrowright X_\omega$ (only up to homothety)
- by hard work of others, X_ω as a space does not depend on ω; but the action does.
- there is a dependence on i (up to bilipschitz equivalence)

- convex-cocompactness is necessary for $\lambda_n \to \infty$; $\exists$ surface subgps of RAAGs without this property.

3) $X_\omega \hookrightarrow \prod_{v \in \Gamma} T^v_\omega$ G-equivariant isometric embedding with T^v_ω an $\mathbb{R}$-tree.

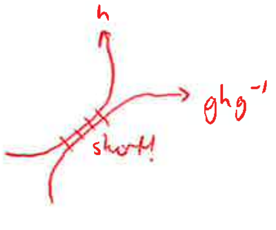
nb: choices coming from X_n, not from a Bestvina-Paulin construction on T^v.

but: need (at least!) some niceness of arc stabilisers to do anything here.

To understand $G \curvearrowright T^v_\omega$, go back to $G \curvearrowright T^v$.

weakly weakly properly discontinuous

Def an action $H \curvearrowright T$ on a simplicial is <u>uniformly WWPD</u> if $\exists K \geq 1$ st
$\forall g, h$ ~~with~~ $\in H$ with h loxodromic in T :
- either $g \cdot Ax(h;T) = Ax(h;T)$
- or $|g \cdot Ax(h;T) \cap Ax(h,T)| \leq K \cdot \ell(h,T)$



= Delzant's "weak acylindricity"

Fact 1:

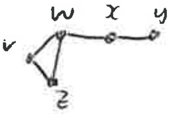
- each of the trees $G \curvearrowright T^v$, $v \in \Gamma$, is uniformly WWPD
- if $h \in G$ is loxodromic in T^v and $\langle h \rangle$ is convex-cocompact in $\mathcal{X}_\Gamma$, then $\mathrm{Stab}_G(Ax(h, T^v))$ is $Z_G(h)$. ($\geq$ always)

Fact 2

$\exists\, c \geq 1$ st $\forall\, \beta \subseteq Min(G; T^v)$ arc with $|\beta| > c$ $\exists\, \beta_0 \subseteq \beta$ a subarc with $|\beta_0| \geq |\beta| - c$ st β_0 satisfies a dichotomy:

(a) the G-stabiliser of β_0 is a centraliser in G

(b) $\exists\, g_0 \in G$ st $\beta_0 \subseteq Ax(g_0; T^v)$ and $\ell(g; \mathcal{X}_\Gamma) \leq c$.

eg $G = A_\Gamma$, $\Gamma =$ 

For RAAGs: can always take $\beta_0 = \beta$, but still see dichotomy

Consider the arcs of T^v obtained as "shadows" of the following words in A_Γ:

- v^{n+1} : arc stabiliser is $A_{lk(v)} \cap v^n A_{lk(v)} v^{-n} = \langle w, z \rangle$ option (b): not a centraliser
- $(vx)^{n+1}$: here $A_{lk(v)} \cap (vx)^n A_{lk(v)} (vx)^{-n} = \langle w \rangle$ option (a): $\langle w \rangle = Z_{A_\Gamma}(\langle v, x \rangle)$

$\rightsquigarrow$ centralisers are to do with intersections of stars, which behave nicely; intersections of links are not quite so well behaved.

Def a baby line is a line $\alpha \subset T^v_\omega$ obtained as $\lim_\omega Ax(g_n, T^v)$ for loxodromics $g_n \in G$ with $\lim_\omega \frac{1}{\lambda_n} \ell(g_n; T^v) = 0$

"much like babies, they are kind of a pain in the neck. But with time you learn they are quite useful"

notation: β arc $\rightsquigarrow$ $G_\beta = \{ g \in G \mid gx = x,\ \forall x \in \beta \}$ (pointwise)

α line $\rightsquigarrow$ $G_\alpha = \{ g \in G \mid g\alpha = \alpha \}$ (set-wise)

(though no arc inversions in our trees)

Thm consider $G \curvearrowright \mathcal{X}_\omega$ degeneration and $v \in T$ st $G \curvearrowright T^v_\omega$ not elliptic.

(1) if an arc $\beta \subseteq Min(G; T^v_\omega)$ is not contained in any baby lines, then G_β is a centraliser in G

(2) distinct baby lines share at most one point

(3) $\alpha \subseteq T^v_\omega$ baby line $\Rightarrow$ G_α is a centraliser in G acting on α via a homomorphism $\rho_\alpha : G_\alpha \rightarrow \mathbb{R}$. Moreover $\forall\, \beta \subseteq \alpha$ arc, $G_\beta = \ker(\rho_\alpha)$.

"trying to advertise the friendlier side of things" proof/

1) $\cdot$ β are $\notin$ baby line. $\overset{\text{Fact 2}}{\rightsquigarrow}$ approximate by $\beta_n \subseteq T^v$ satisfying (a).

- $G_\beta \overset{?}{=} \bigcap_\omega \varphi_n^{-1}(G_{\beta_n}.)$ is a centraliser

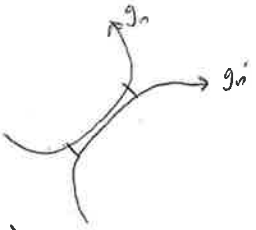
→ centralisers are invariant under automorphisms and intersections, and we are using this fact

$\geq$ straightforward

$\leq$ because $\beta \notin$ baby lines.

2) $\alpha \cap \alpha' \geq \beta$

$$\underset{\underset{\alpha_n}{\shortparallel}}{\overset{\shortparallel}{\lim Ax(g_n)}} \qquad \underset{\underset{\alpha_n'}{\shortparallel}}{\overset{\shortparallel}{\lim Ax(g_n')}}$$



$$|g_n'\alpha_n \cap \alpha_n| \gg \ell(g_n; T^v)$$

$$\Downarrow \text{WWPD}$$

$$g_n'\alpha_n = \alpha_n \Rightarrow \alpha = \alpha'$$

3) [cut for time]

Fact 3: $\alpha \subseteq T^v_\omega$ baby line of $G_\alpha \curvearrowright \alpha$ non-trivially, then (both)

- G_α is a standard virtual product
- we can write $G = G_\alpha \underset{\mathrm{Tor}(\beta_\alpha)}{*} H$ for some subgp H.

(not immediate, even from the transverse family: need to approximate the $\mathbb{R}$-tree by a geometric $\mathbb{R}$-tree fol.)

Def a standard virtual product is a subgp of G obtained as follows

- start with a maximal subgp of G virtually splitting as a direct product (called P).

  either stop or continue:

- write a finite index subgp of P as $P_1 \times \dots \times P_k \times \mathbb{Z}^n$ with P_i not a virtual product, then replace P_1 with a maximal virtual product $P_1' \leq P_1$, and take the maximal finite index overgroup of $P_1' \times P_2 \times \dots \times P_k \times \mathbb{Z}^n$

  either stop or continue.

Towards finite generation ...

Def H-gp. A Dehn twist is some $\tau \in Aut(H)$ arising from a 1-edge splitting of H as follows:

- $H = A *_C B$, $z \in Z_B(C) \rightsquigarrow \tau : \begin{cases} \tau|_A = id_A \\ \tau|_B = ad_z \end{cases}$

eg. - classical Dehn twists of ~~RAAGs~~ surfaces wrt separating curves

- partial conjugations of RAAGs

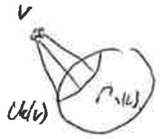
- $H = A *_{C,\gamma} = \langle A, t \mid tct^{-1} = \gamma(c), \forall c \in C \rangle$

$C \leq A$, $\gamma : C \hookrightarrow A$.

$\tau|_A = id$ and $\begin{cases} \tau(t) = tz & \text{for } z \in Z_A(C) \\ \tau(t) = zt & \text{for } z \in Z_A(\gamma(C)). \end{cases}$

eg. - Dehn twists on non-separating curves

- transvections in RAAGs.



Rem up to folding the splitting, can always assume $z \in$ centre (C).

eg. $G = A *_C B \rightsquigarrow G = \underbrace{A * Z_B(z)}_{A'} *_{Z_B(z)} B$.

(but this makes the original structure less visible)

Thm G special $\Rightarrow Out(G)$ is f.g.

Fix $i : G \hookrightarrow A_\Gamma$ convex cocompact embedding and ω non-principal ultrafilter

Shortening Let $F \subseteq G$ finite subset. Let $\varphi_n, \psi_n \in Aut(G)$ be sequences of automorphisms

Write $(\varphi_n) \underset{F}{\ll} (\psi_n)$ (resp. $\underset{F}{\simeq}$)

if $\min_{x \in X_n} \max_{f \in F} d(x, \varphi_n(f)x) < \min_{x \in X_n} \max_{f \in F} d(x, \psi_n(f)x)$ for ω-all n

(resp. $=$ for ω-all n).

Our goal: let $S \subseteq G$ be a finite generating set; $\varphi_n \in$ ~~Out~~ $\mathrm{Aut}(G)$ be automorphisms in distinct outer classes.

Then $\exists\, G \curvearrowright T$ 1-edge splitting and $\exists\, \tau_n \in \mathrm{Aut}(G)$ Dehn twists wrt T st

$$(\varphi_n \tau_n) \underset{S}{<<} (\varphi_n).$$

[nb: this will fail in general, but is nevertheless the goal/strategy]

Strategy

asymptotic cone of X_n

(φ_n) sequence of automorph $\rightsquigarrow$ $\rho : G \to \mathrm{Isom}(X_\omega)$

(λ_n scaling factors)

$\Downarrow$

pick $v \in \Gamma$ st $G \curvearrowright T^v_\omega$ is non-elliptic

$\Downarrow$

produce a 1-edge splitting $G \curvearrowright T$ and Dehn twists τ_n wrt T

Let $\hat{\rho} : G \to \mathrm{Isom}(X_\omega)$ obtained from the sequence $(\varphi_n \tau_n)$ with the "old" scaling factors.

Hope: find the τ_n st $\min\limits_{X_\omega} \max\limits_{S}$ ~~$d($~~ $d(x, \hat{\rho}(s)x) < \min\limits_{X_\omega} \max\limits_{s} d(x, \rho(s)x)$

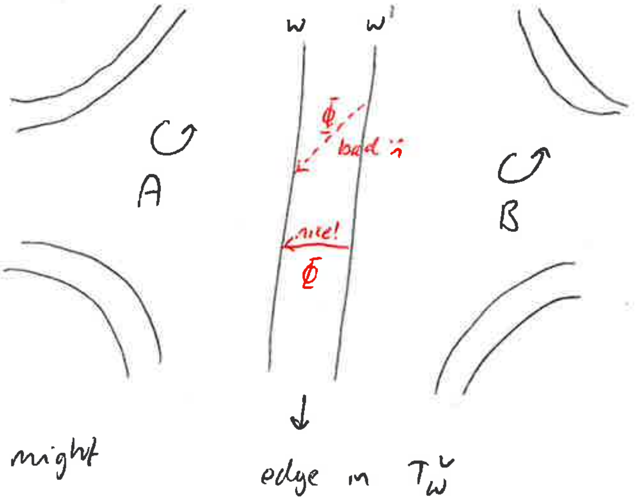
(if you this too much at random it might instead by infinite, so don't do that - details not provided - but if you do very well they go to 0 and that's excellent)

Key example: suppose that $\mathrm{Min}(G; T^v_\omega)$ is a simplicial (1-edge splitting)

$G = A *_C B$

Let $\Phi \in \mathrm{Isom}(X_\omega)$ satisfy $\Phi(w') = w$ and $[\Phi, \rho(c)] = 1$

Suppose that $\begin{cases} \hat{\rho} = \rho \text{ on } A \\ \hat{\rho} = \mathrm{ad}_{\Phi} \circ \rho \text{ on } B \end{cases}$



If Φ "shears" along w (dotted) then $\hat{\rho}(S)$ might be "longer" than $\rho(S)$.

Fact 4: Let $G \curvearrowright X_\omega$ degeneration arising from (φ_n).

Then $\exists\, G \curvearrowright T$ and $\exists\, \tau_n$ Dehn twists, wrt T such that $(\varphi_n \tau_n) \underset{S}{<<} (\varphi_n)$ provided that, for some $v \in \Gamma$ with $G \curvearrowright T^v_\omega$ not elliptic, either:

(a) $\exists\ \beta \subseteq \mathrm{Min}(G, T^v_\omega)$ arc st $N_G(G_\beta)$ is elliptic in T^v_ω ;

(b) or $\exists\ \alpha \in \mathrm{Min}(G; T^v_\omega)$ baby line st $\mathrm{centre}(G_\alpha)$ is elliptic in T^v_ω.

- it seems difficult to both get a multiplier which doesn't shear or which is elliptic (+ so gives you an honest Dehn twist); either property individually is not so bad?

There is an $\mathrm{Aut}(G)$-invariant collection of finitely many conjugacy classes of subgps of G

$[H_1], \dots, [H_k]$ "the hierarchy".

→ choose ordering so $H_i \le H_j \Rightarrow i \le j$

choose $S_i \le H_i$ finite generating set.

Revised goal: $\forall\ (\varphi_n)$, find T, τ_n, st $\exists\, i$ with $(\varphi_n \tau_n) \underset{S_i}{<<} (\varphi_n)$

$(\varphi_n \tau_n) \underset{S_j}{\simeq} (\varphi_n) \quad \forall\ 1 \le j < i$.

ie, shorten as soon as possible + don't make it worse earlier.

rough idea: construct $G \curvearrowright X_\omega$ and pick least i st H_i is not elliptic in X_ω.

Then pick $v \in \Gamma$ st H_i is not elliptic in T^v_ω

Now shorten H_i by Dehn twists of H_i that extend to G. (and are inner on all H_j, $j<i$.)

Rem baby lines with non-elliptic centre can still stop us

(shortening to virtually f.g. by Dehn twists is a classical argument of Rips-Sela).

Pf of "can always shorten by Dehn twists" $\Rightarrow$ $\mathrm{Out}(G)$ is virt. generated by Dehn twists

$\Delta \le \mathrm{Out}(G)$. Say (towards a contradiction) that Δ has inf. index.

$\shortparallel$

$\langle$Dehn twists$\rangle$

Choose coset reps $\Phi_n \in \mathrm{Out}(G)$ st Φ_n is $<<$-shortest in $\Delta\Phi_n$

(important: $<<$ must be a well-ordering on $\mathrm{Out}(G)$)

Now, Φ_n give degeneration $G \curvearrowright X_\omega$ and by the goal there are some Dehn twists $\tau_n \in \Delta$ st $\Phi_n \tau_n << \Phi_n$ ↯ □

Q: where do the other generators come ~~from~~ from?

splitting $G = G_\alpha \underset{K_\alpha}{*} H$

$\uparrow$ not Aut-inv, though G_α is.

$$\rightsquigarrow \mathrm{Out}^0(G) \to \mathrm{Out}(G_\alpha) \to \underset{\substack{\cong \\ \mathbb{Z}^2}}{\mathrm{Out}(G_\alpha / K_\alpha)}$$

+ the problem arises from the fact that $GL_2\mathbb{Z} = \mathrm{Out}(\mathbb{Z}^2)$ is not virt. generated by high powers of elem. matrices.

nb not a problem for RAAGs because all virtual products are honest products.

need to use the fact (from Hongbin Sun) that there are special gps with arbitrarily much torsion in their homology to build the bad examples.